2 The z and Fourier transforms

for discrete time signals

Continuous time: Laplace transform, and (C.T.)Fourier transform as special case
Discrete time: z transform, and (D.T.) Fourier transform as special case

2.1 Introduction

Laplace transf.: 2D function of a 2D variable,sometimes less practical,
but converges for more signals than Fourier (e.g. unstable systems/signals).
Fourier transf.: 2D function of a 1D variable, and |X(w)| is a 1D function
of a 1D variable, easier to use (or visualize) for some applications

In Chapter 1, we studied linear time-invariant systems, using both impulse responses and difference equations to characterize them. In this chapter, we study another very useful way to characterize discrete-time systems. It is linked with the fact that, when an exponential function is input to a linear time-invariant system, its output is an exponential function of the same type, but with a different amplitude. This can be deduced by considering that, from equation (1.38), a linear time-invariant discrete-time system with impulse response $h(n)$, when excited by an exponential $x(n) = z^n$, produces at its output a signal $y(n)$ such that

"eigenfunction"

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) = \sum_{k=-\infty}^{\infty} z^{n-k}h(k) = z^n \sum_{k=-\infty}^{\infty} h(k)z^{-k} \tag{2.1}$$

that is, the signal at the output is also an exponential z^n, but with an amplitude multiplied by the complex function

$$H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k} \tag{2.2}$$

simple scalar,
depends on z
value

In this chapter, we characterize linear time-invariant systems using the quantity $H(z)$ in the above equation, commonly known as the z transform of the discrete-time sequence $h(n)$. As we will see later in this chapter, with the help of the z transform, linear convolutions can be transformed into simple algebraic equations. The importance of this for discrete-time systems parallels that of the Laplace transform for continuous-time systems.

The case when z^n is a complex sinusoid with frequency ω, that is, $z = e^{j\omega}$, is of particular importance. In this case, equation (2.2) becomes

special case where z magnitude is unity

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{2.3}$$

which can be represented in polar form as $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\Theta(\omega)}$, yielding, from equation (2.1), an output signal $y(n)$ such that

$$y(n) = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\Theta(\omega)}e^{j\omega n} = |H(e^{j\omega})|e^{j\omega n + j\Theta(\omega)} \tag{2.4}$$

change of magnitude and phase,
same for cos(wn) and sin(wn) functions, since they
can be expressed in terms of exp(jwn).

This relationship implies that the effect of a linear system characterized by $H(e^{j\omega})$ on a complex sinusoid is to multiply its amplitude by $|H(e^{j\omega})|$ and to add $\Theta(\omega)$ to its phase. For this reason, the descriptions of $|H(e^{j\omega})|$ and $\Theta(\omega)$ as functions of ω are widely used to characterize linear time-invariant systems, and are known as their magnitude and phase responses, respectively. The complex function $H(e^{j\omega})$ in equation (2.4) is also known as the Fourier transform of the discrete-time sequence $h(n)$. The Fourier transform is as important for discrete-time systems as it is for continuous-time systems.

In this chapter, we will study the z and Fourier transforms for discrete-time signals. We begin by defining the z transform, discussing issues related to its convergence and its relation to the stability of discrete-time systems. Then we present the inverse z transform, as well as several z-transform properties. Next, we show how to transform discrete-time convolutions into algebraic equations, and introduce the concept of a transfer function. We then present an algorithm to determine, given the transfer function of a discrete-time system, whether the system is stable or not and go on to discuss how the frequency response of a system is related to its transfer function. At this point, we give a formal definition of the Fourier transform of discrete-time signals, highlighting its relations to the Fourier transform of continuous-time signals. An expression for the inverse Fourier transform is also presented. Its main properties are then shown as particular cases of those of the z transform. We close the chapter by presenting some MATLAB functions which are related to z and Fourier transforms, and which aid in the analysis of transfer functions of discrete-time systems.

2.2 Definition of the z transform

The z transform of a sequence $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2.5}$$

where z is a complex variable. Note that $X(z)$ is only defined for the regions of the complex plane in which the summation on the right converges. → ROC, "region of convergence"

Very often, the signals we work with start only at $n = 0$, that is, they are nonzero only for $n \geq 0$. Because of that, some textbooks define the z transform as

$$X_U(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.6}$$

→ mostly useful for solving difference equations with non-zero initial conditions, like the unilateral Laplace transform

unilateral

which is commonly known as the one-sided z transform, while equation (2.5) is referred to as the two-sided z transform. Clearly, if the signal $x(n)$ is nonzero for $n < 0$, then the one-sided and two-sided z transforms are different. In this text, we work only with the two-sided z transform, which is referred to, without any risk of ambiguity, just as the z transform.

bilateral

As mentioned above, the z transform of a sequence exists only for those regions of the complex plane in which the summation in equation (2.5) converges. The example below clarifies this point.

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad \text{si } |a| < 1$$

$$\sum_{k=n_1}^{\infty} a^k = \frac{a^{n_1}}{1-a} \quad \text{si } |a| < 1$$

$$\sum_{k=0}^{n_1} a^k = \frac{1-a^{n_1+1}}{1-a} \quad \text{si a} \neq 1$$

$$\sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a} \quad \text{si a} \neq 1$$

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad \text{si } |a| < 1$$

EXAMPLE 2.1

Compute the z transform of the sequence $x(n) = K u(n)$.

SOLUTION

By definition, the z transform of $K u(n)$ is

$$X(z) = K \sum_{n=0}^{\infty} z^{-n} = K \sum_{n=0}^{\infty} (z^{-1})^n$$

Thus, $X(z)$ is the sum of a power series which converges only if $|z^{-1}| < 1$. case, $X(z)$ can be expressed as

$$X(z) = \frac{K}{1 - z^{-1}} = \frac{K z}{z - 1}, \quad |z| > 1 \tag{2.8}$$

Note that for $|z| < 1$, the nth term of the summation, z^{-n}, tends to infinity as $n \to \infty$, and therefore $X(z)$ is not defined. For $z = 1$, the summation is also infinite. For $z = -1$, the summation oscillates between 1 and 0. In none of these cases does the z transform converge.

$\triangle$

It is important to note that the z transform of a sequence is a Laurent series in the complex variable z (Churchill, 1975). Therefore, the properties of Laurent series apply directly to the z transform. As a general rule, we can apply a result from series theory stating that, given a series of the complex variable z

$$S(z) = \sum_{i=0}^{\infty} f_i(z) \tag{2.9}$$

such that $|f_i(z)| < \infty$, $i = 0, 1, \ldots$, and given the quantity

$$\alpha(z) = \lim_{n\to\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| \tag{2.10}$$

then the series converges absolutely if $\alpha(z) < 1$, and diverges if $\alpha(z) > 1$ (Kreyszig, 1979). Note that, for $\alpha(z) = 1$, the above procedure tells us nothing about the convergence of the series, which must be investigated by other means. One can justify this by noting that, if $\alpha(z) < 1$, the terms of the series are under an exponential a^n for some $a < 1$, and therefore their sum converges as $n \to \infty$. One should clearly note that, if $|f_i(z)| = \infty$, for some i, then the series is not convergent.

The above result can be extended for the case of two-sided series as in the equation below

$$S(z) = \sum_{i=-\infty}^{\infty} f_i(z) \tag{2.11}$$

if we express $S(z)$ above as the sum of two series $S_1(z)$ and $S_2(z)$ such that

$$S_1(z) = \sum_{i=0}^{\infty} f_i(z) \quad \text{and} \quad S_2(z) = \sum_{i=-\infty}^{-1} f_i(z) \tag{2.12}$$

then $S(z)$ converges if the two series $S_1(z)$ and $S_2(z)$ converge. Therefore, in this case, we have to compute the two quantities

$$\alpha_1(z) = \lim_{n\to\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| \quad \text{and} \quad \alpha_2(z) = \lim_{n\to-\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| \tag{2.13}$$

Naturally, $S(z)$ converges absolutely if $\alpha_1(z) < 1$ and $\alpha_2(z) > 1$. The condition $\alpha_1(z) < 1$ is equivalent to saying that, for $n \to \infty$, the terms of the series are under a^n for some $a < 1$. The condition $\alpha_2(z) > 1$ is equivalent to saying that, for $n \to -\infty$, the terms of the series are under b^n for some $b > 1$. One should note that, for convergence, we must also have $|f_i(z)| < \infty, \forall i$.

Applying these convergence results to the z-transform definition given in equation (2.5), we conclude that the z transform converges if

$$\alpha_1 = \lim_{n\to\infty} \left| \frac{x(n+1)z^{-n-1}}{x(n)z^{-n}} \right| = \left|z^{-1}\right| \lim_{n\to\infty} \left| \frac{x(n+1)}{x(n)} \right| < 1 \tag{2.14}$$

$$\alpha_2 = \lim_{n\to-\infty} \left| \frac{x(n+1)z^{-n-1}}{x(n)z^{-n}} \right| = \left|z^{-1}\right| \lim_{n\to-\infty} \left| \frac{x(n+1)}{x(n)} \right| > 1 \tag{2.15}$$

Defining

$$r_1 = \lim_{n\to\infty} \left| \frac{x(n+1)}{x(n)} \right| \tag{2.16}$$

$$r_2 = \lim_{n\to-\infty} \left| \frac{x(n+1)}{x(n)} \right| \tag{2.17}$$

then equations (2.14) and (2.15) are equivalent to

$$r_1 < |z| < r_2 \tag{2.18}$$

That is, the z transform of a sequence exists in an annular region of the complex plane defined by equation (2.18) and illustrated in Figure 2.1. It is important to note that, for some sequences, $r_1 = 0$ or $r_2 \to \infty$. In these cases, the region of convergence may or may not include $z = 0$ or $|z| = \infty$, respectively.

We now take a closer look at the convergence of z transforms for four important classes of sequences.

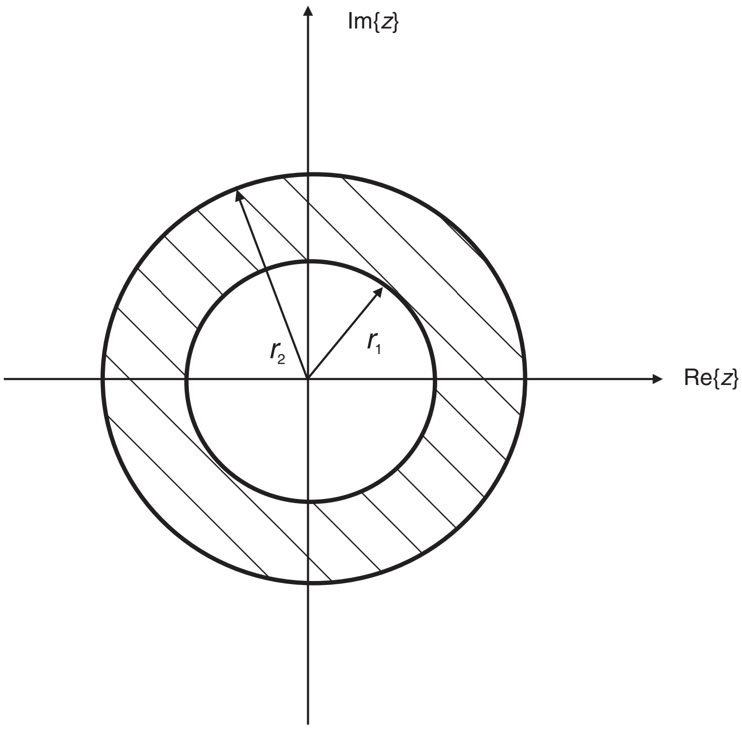

Figure 2.1 General region of convergence of the z transform.

- Right-handed, one-sided sequences: These are sequences such that $x(n) = 0$, for $n < n_0$, that is

  includes causal signals (n0>=0)

$$X(z) = \sum_{n=n_0}^{\infty} x(n)z^{-n} \tag{2.19}$$

  In this case, the z transform converges for $|z| > r_1$, where r_1 is given by equation (2.16). Since $|x(n)z^{-n}|$ must be finite, then, if $n_0 < 0$, the convergence region excludes $|z| = \infty$.
- Left-handed, one-sided sequences: These are sequences such that $x(n) = 0$, for $n > n_0$, that is

  includes anti-causal signals (no<=0)

$$X(z) = \sum_{n=-\infty}^{n_0} x(n)z^{-n} \tag{2.20}$$

  In this case, the z transform converges for $|z| < r_2$, where r_2 is given by equation (2.17). Since $|x(n)z^{-n}|$ must be finite, then, if $n_0 > 0$, the convergence region excludes $|z| = 0$.
- Two-sided sequences: In this case,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2.21}$$

and the z transform converges for $r_1 < |z| < r_2$, where r_1 and r_2 are given by equations (2.16) and (2.17). Clearly, if $r_1 > r_2$, then the z transform does not exist.

- Finite-length sequences: These are sequences such that $x(n) = 0$, for $n < n_0$ and $n > n_1$, that is

$$X(z) = \sum_{n=n_0}^{n_1} x(n)z^{-n} \tag{2.22}$$

In such cases, the z transform converges everywhere except at the points such that $|x(n)z^{-n}| = \infty$. This implies that the convergence region excludes the point $z = 0$ if $n_1 > 0$ and $|z| = \infty$ if $n_0 < 0$.

EXAMPLE 2.2

Compute the z transforms of the following sequences, specifying their region of convergence:

(a) $x(n) = k2^n u(n)$
(b) $x(n) = u(-n + 1)$
(c) $x(n) = -k2^n u(-n - 1)$
(d) $x(n) = 0.5^n u(n) + 3^n u(-n)$
(e) $x(n) = 4^{-n} u(n) + 5^{-n} u(n + 1)$

SOLUTION

(a) $X(z) = \sum_{n=0}^{\infty} k2^n z^{-n}$

This series converges if $|2z^{-1}| < 1$, that is, for $|z| > 2$. In this case, $X(z)$ is the sum of a geometric series, and therefore

$$X(z) = \frac{k}{1 - 2z^{-1}} = \frac{kz}{z - 2}, \quad \text{for } 2 < |z| \leq \infty \tag{2.23}$$

(b) $X(z) = \sum_{n=-\infty}^{1} z^{-n}$

This series converges if $|z^{-1}| > 1$, that is, for $|z| < 1$. Also, in order for the term z^{-1} to be finite, $|z| \neq 0$. In this case, $X(z)$ is the sum of a geometric series, such that

$$X(z) = \frac{z^{-1}}{1 - z} = \frac{1}{z - z^2}, \quad \text{for } 0 < |z| < 1 \tag{2.24}$$

(c) $X(z) = \sum_{n=-\infty}^{-1} -k2^n z^{-n}$

This series converges if $|\frac{z}{2}| < 1$, that is, for $|z| < 2$. In this case, $X(z)$ is the sum of a geometric series, such that

$$X(z) = \frac{-k\frac{z}{2}}{1-\frac{z}{2}} = \frac{kz}{z-2}, \quad \text{for } 0 \leq |z| < 2 \tag{2.25}$$

(d) $X(z) = \sum_{n=0}^{\infty} 0.5^n z^{-n} + \sum_{n=-\infty}^{0} 3^n z^{-n}$

This series converges if $|0.5z^{-1}| < 1$ and $|3z^{-1}| > 1$, that is, for $0.5 < |z| < 3$. In this case, $X(z)$ is the sum of two geometric series, and therefore

$$X(z) = \frac{1}{1-0.5z^{-1}} + \frac{1}{1-\frac{1}{3}z} = \frac{z}{z-0.5} + \frac{3}{3-z}, \quad \text{for } 0.5 < |z| < 3 \tag{2.26}$$

(e) $X(z) = \sum_{n=0}^{\infty} 4^{-n} z^{-n} + \sum_{n=-1}^{\infty} 5^{-n} z^{-n}$

This series converges if $|\frac{1}{4}z^{-1}| < 1$ and $|\frac{1}{5}z^{-1}| < 1$, that is, for $|z| > \frac{1}{4}$. Also, the term for $n = -1$, $\left(\frac{1}{5}z^{-1}\right)^{-1} = 5z$, is finite only for $|z| < \infty$. In this case, $X(z)$ is the sum of two geometric series, resulting in

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{5z}{1-\frac{1}{5}z^{-1}} = \frac{4z}{4z-1} + \frac{25z^2}{5z-1}, \quad \text{for } \frac{1}{4} < |z| < \infty \tag{2.27}$$

In this example, although the sequences in items (a) and (c) are distinct, the expressions for their z transforms are the same, the difference being only in their regions of convergence. This highlights the important fact that, in order to completely specify a z transform, its region of convergence must be supplied. In Section 2.3, when we study the inverse z transform, this issue is dealt with in more detail.

△

In several cases we deal with causal and stable systems. Since for a causal system its impulse response $h(n)$ is zero for $n < n_0$, then, from equation (1.48), we have that a causal system is also BIBO stable if

$$\sum_{n=n_0}^{\infty} |h(n)| < \infty \tag{2.28}$$

Applying the series convergence criterion seen above, we have that the system is stable only if

$$\lim_{n\to\infty} \left| \frac{h(n+1)}{h(n)} \right| = r < 1 \tag{2.29}$$

for causality, ROC is outside

This is equivalent to saying that $H(z)$, the z transform of $h(n)$, converges for $|z| > r$. Since, for stability, $r < 1$, then we conclude that the convergence region of the z

for stability, ROC includes the unit circle for causality and stability:

transform of the impulse response of a stable causal system includes the region outside the unit circle and the unit circle itself (in fact, if $n_0 < 0$, then this region excludes $|z| = \infty$).

A very important case is when $X(z)$ can be expressed as a ratio of polynomials, in the form

rational form

$$X(z) = \frac{N(z)}{D(z)} \tag{2.30}$$

We refer to the roots of $N(z)$ as the zeros of $X(z)$ and to the roots of $D(z)$ as the poles of $X(z)$. More specifically, in this case $X(z)$ can be expressed as

$$X(z) = \frac{N(z)}{\prod_{k=1}^{K}(z - p_k)^{m_k}} \tag{2.31}$$

where p_k is a pole of multiplicity m_k, and K is the total number of distinct poles. Since $X(z)$ is not defined at its poles, its convergence region must not include them. Therefore, given $X(z)$ as in equation (2.31), there is an easy way of determining its convergence region, depending on the type of sequence $x(n)$:

- Right-handed, one-sided sequences: The convergence region of $X(z)$ is $|z| > r_1$. Since $X(z)$ is not convergent at its poles, then its poles must be inside the circle $|z| = r_1$ (except for poles at $|z| = \infty$), and $r_1 = \max_{1\le k\le K}\{|p_k|\}$. This is illustrated in Figure 2.2a.
- Left-handed, one-sided sequences: The convergence region of $X(z)$ is $|z| < r_2$. Therefore, its poles must be outside the circle $|z| = r_2$ (except for poles at $|z| = 0$), and $r_2 = \min_{1\le k\le K}\{|p_k|\}$. This is illustrated in Figure 2.2b.
- Two-sided sequences: The convergence region of $X(z)$ is $r_1 < |z| < r_2$, and therefore some of its poles are inside the circle $|z| = r_1$ and some outside the circle $|z| = r_2$. In this case, the convergence region needs to be further specified. This is illustrated in Figure 2.2c.

For X(z) with N poles of different radius, there are N+1 possible ROC in general, with N+1 corresponding signals x(n). But only one x(n) signal will be right-sided: the one with the ROC outside all poles. Also, only one x(n) signal can be stable (or "absolutely summable"): the one with ROC including the unit circle.

2.3 Inverse z transform

Very often one needs to determine which sequence corresponds to a given z transform. A formula for the inverse z transform can be obtained from the residue theorem, which we state next.

THEOREM 2.1 (RESIDUE THEOREM)

Let $X(z)$ be a complex function that is analytic inside a closed contour C, including

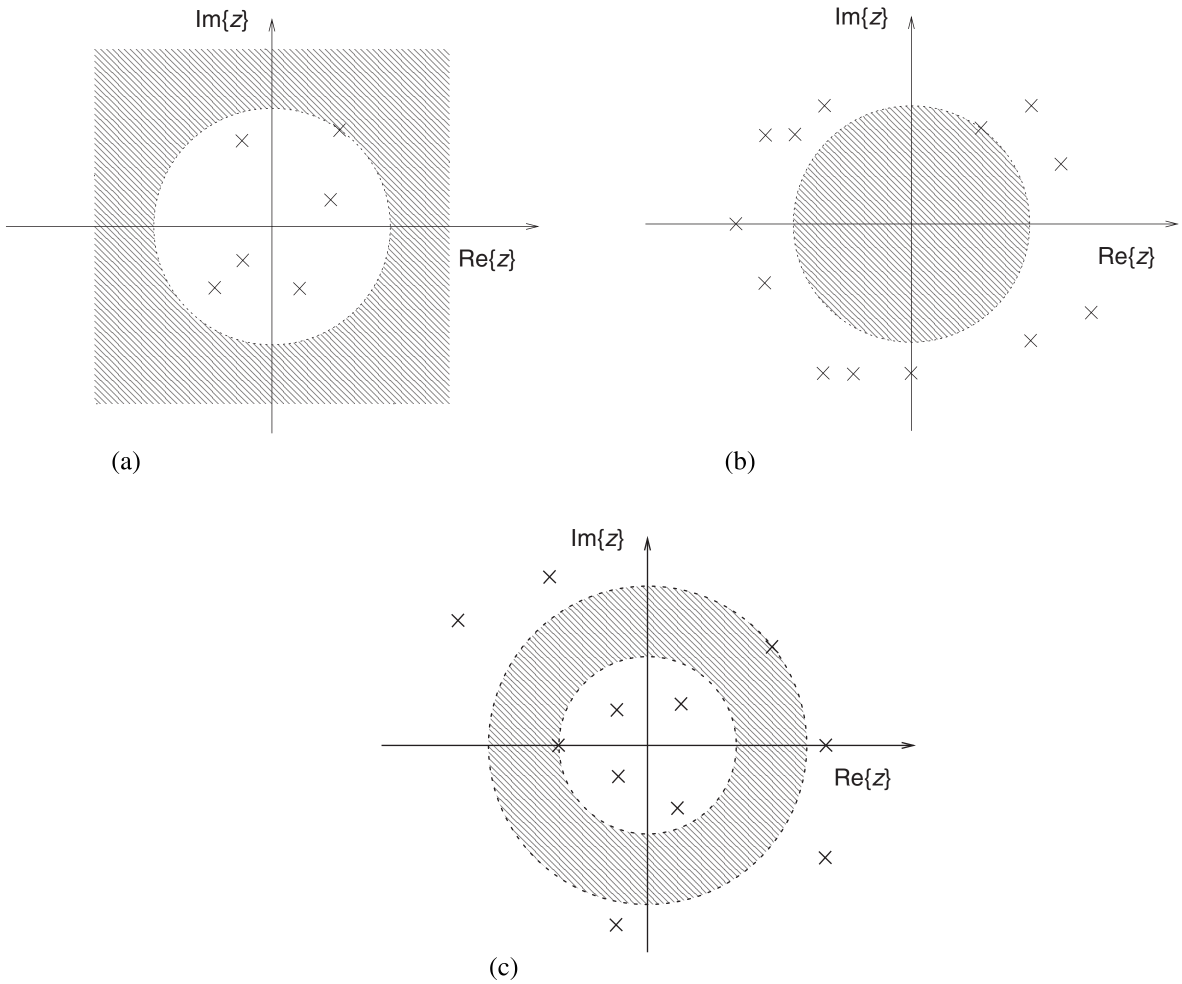

Figure 2.2 Regions of convergence of a z transform in relation to its poles: (a) right-handed, one-sided sequences; (b) left-handed, one-sided sequences; (c) two-sided sequences.

the contour itself, except in a finite number of singular points p_n inside C. In this case, the following equality holds:

$$\oint_C X(z)\mathrm{d}z = 2\pi\,\mathrm{j}\sum_{k=1}^{K} \operatorname*{res}_{z=p_k}\{X(z)\} \tag{2.32}$$

with the integral evaluated counterclockwise around C.

If p_k is a pole of multiplicity m_k of $X(z)$, that is, if $X(z)$ can be written as

$$X(z) = \frac{P_k(z)}{(z-p_k)^{m_k}} \tag{2.33}$$

where $P_k(z)$ is analytic at $z = p_k$, then the residue of $X(z)$ with respect to p_k is given by