Heaps

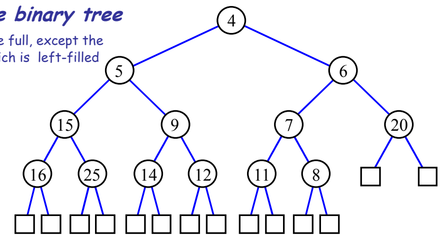
- Heaps
- Properties
- Deletion, Insertion, Construction
- Implementation of the Heap
- Implementation of Priority Queue using a Heap
- An application: HeapSort

Heaps (Min-heap)

Complete binary tree that stores a collection of keys (or key-element pairs) at its internal nodes and that satisfies the additional property:

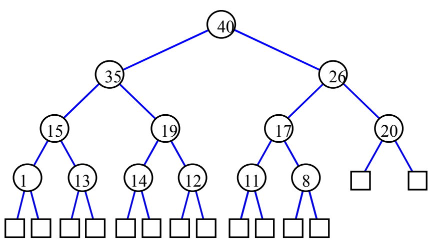
key(parent) ≤ key(child)

REMEMBER:
complete binary tree
all levels are full, except the last one, which is left-filled

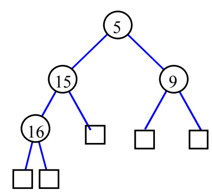


Max-heap

key(parent) ≥ key(child)



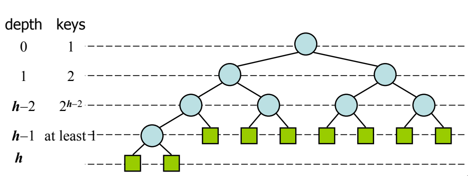
We store the keys in the internal nodes only



After adding the □ leaves the resulting tree is full

Height of a Heap

- Theorem: A heap storing n keys has height $O(\log n)$
  Proof:
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \dots, h - 2$ and at least one key at depth $h - 1$, we have $n \geq 1 + 2 + 4 + \dots + 2^{h-2} + 1$
  - Thus, $n \geq 2^{h-1}$, i.e., $h \leq \log n + 1$



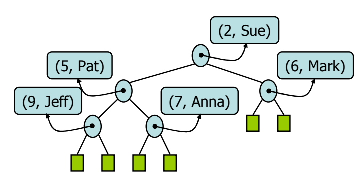
Notice that ….

- We could use a heap to implement a priority queue
- We store a (key, element) item at each internal node

removeMin():

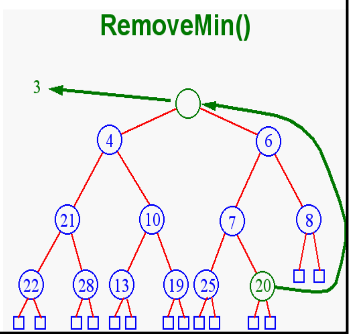
→ Remove the root

→ Re-arrange the heap!

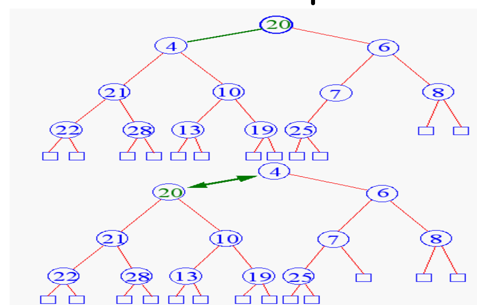


Removal From a Heap

- The removal of the top key leaves a hole
- We need to fix the heap
- First, replace the hole with the last key in the heap
- Then, begin Downheap
…



Downheap



- Downheap compares the parent with the smallest child. If the child is smaller, it switches the two.

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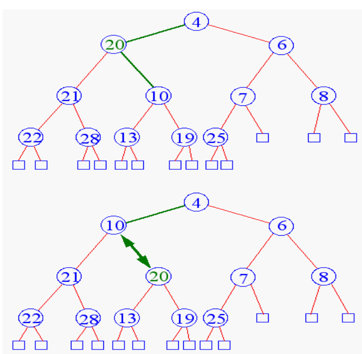
Downheap Continues



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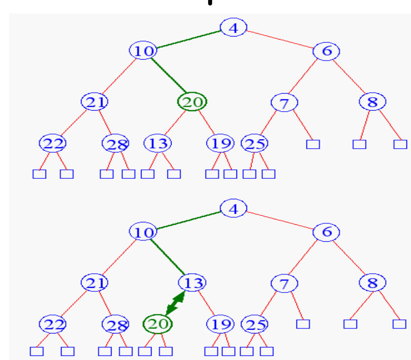
Downheap Continues



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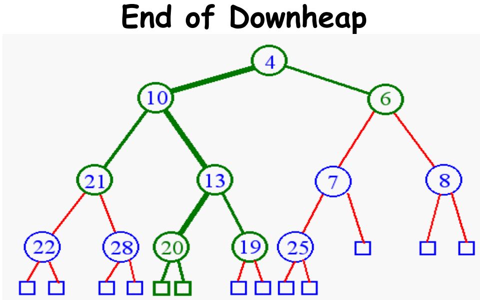
End of Downheap



- Downheap terminates when the key is greater than the keys of both its children or the bottom of the heap is reached.
  - (total #swaps) $\leq$ $(h - 1)$, which is $O(\log n)$

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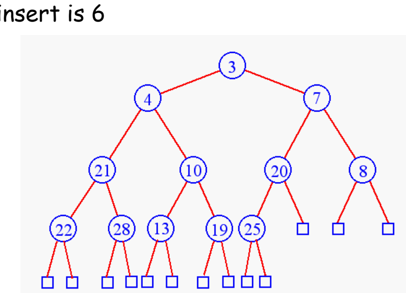
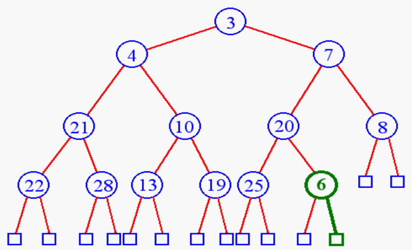
Heap Insertion

The key to insert is 6



Heap Insertion

Add the key in the ***next available position*** in the heap.



Now begin ***Upheap***.

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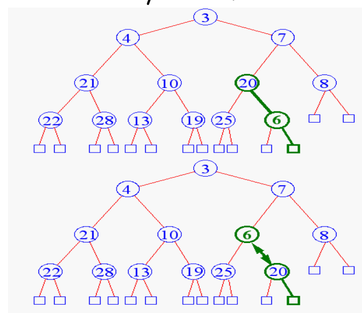
Upheap

- Swap parent-child keys out of order



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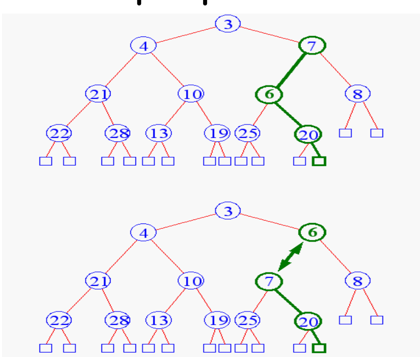
Upheap Continues



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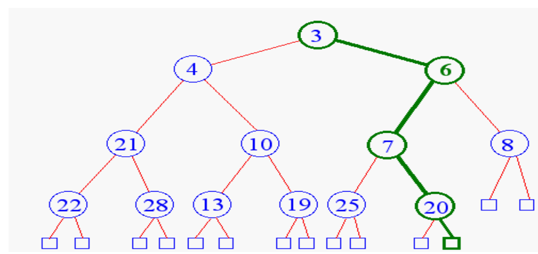
End of Upheap



- Upheap terminates when new key is greater than the key of its parent c≤ the top of the heap is reached
- (total #swaps) (h - 1), which is O(log n)

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Heap Construction

We could insert the Items one at the time with a sequence of Heap Insertions:

$$\sum_{k=1}^{n} \log k = O(n \log n)$$

But we can do better ....

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Bottom-up Heap Construction

- We can construct a heap storing ***n*** given keys using a bottom-up construction

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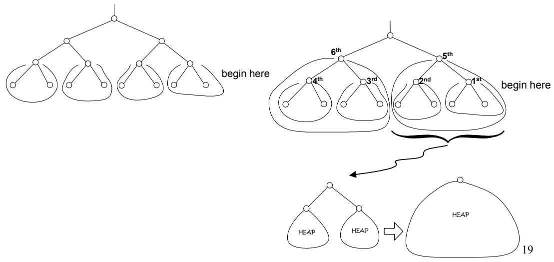
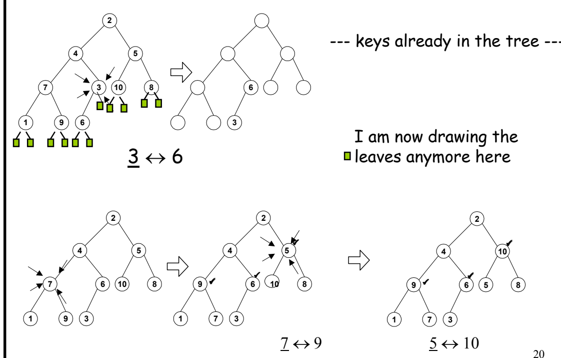
Construction of a Heap

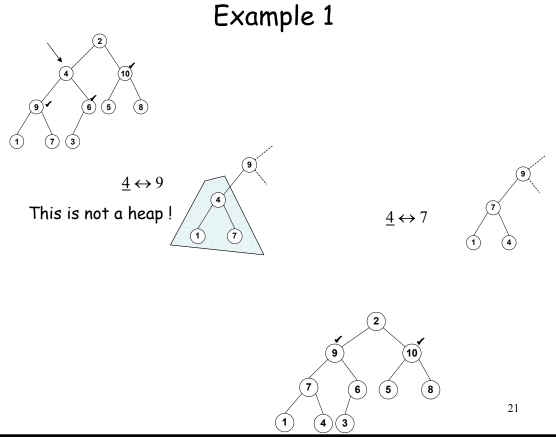
Idea: Recursively re-arrange each sub-tree in the heap starting with the leaves



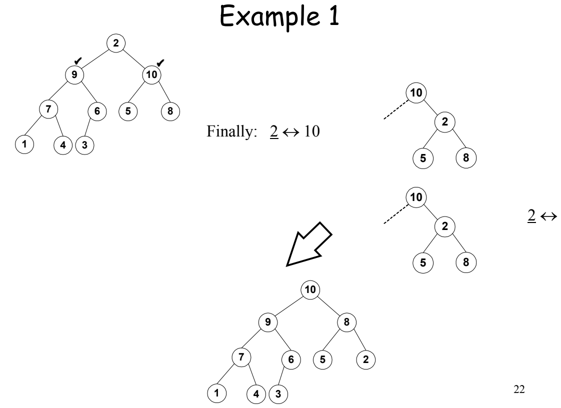
Example 1 (Max-Heap)



Example 1



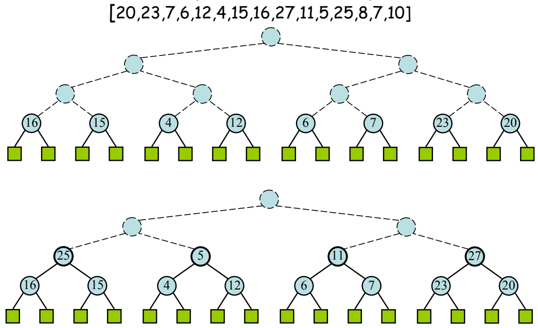
Example 1



--- keys given one at a time ---

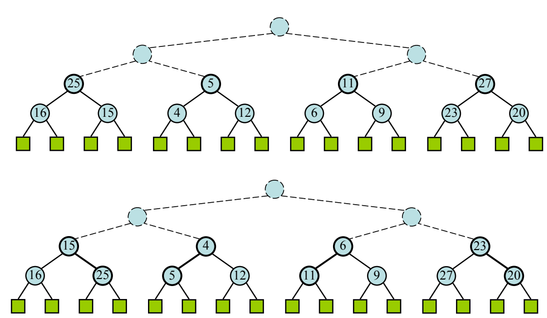
Example 2 (min-heap)

[20,23,7,6,12,4,15,16,27,11,5,25,8,7,10]



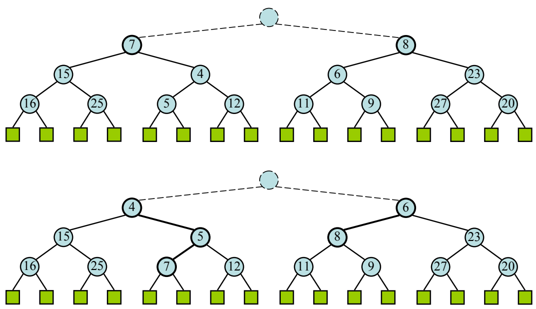
Example 2

20,23,7,6,12,4,15,16,27,11,5,25,8,7,10]



Example 2

20,23,7,6,12,4,15,16,27,11,5,25,8,7,10]



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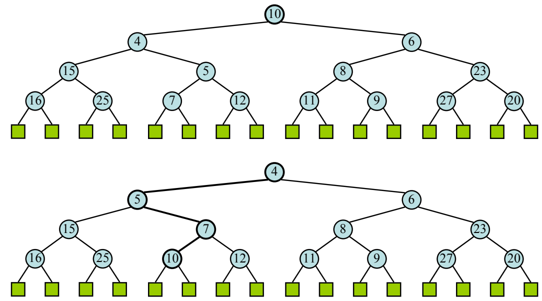
Example 2

20,23,7,6,12,4,15,16,27,11,5,25,8,7,10]



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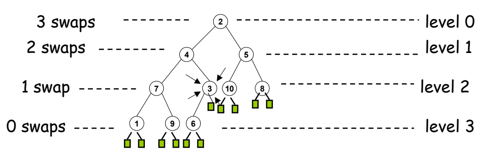
Analysis of Heap Construction

Number of swaps

h = 4



Let L be the max level
(L = h-1)

level i -------- L - i swaps

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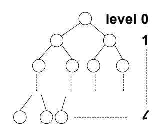
Analysis of Heap Construction

Number of swaps



At level i the number of swaps is

$\leq$ L – i for each node

At level i there are $\leq 2^i$ nodes

Total: $\leq \sum_{i=0}^{\ell}(L-i)\cdot 2^i$

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Calculating $O(\sum(L-i)\cdot 2^i)$

Let j = L-i, then i = L-j and

$\sum_{i=0}^{L}(L-i)\cdot 2^i \quad = \sum_{j=0}^{L} j\, 2^{L-j} = 2^L \sum_{j=0}^{L} j\, 2^{-j}$

Consider $\sum j\cdot 2^{-j}$:

$\sum j\cdot 2^{-j}$ = 1/2 + 2 1/4 + 3 1/8 + 4 1/16 + …
= 1/2 + 1/4 + 1/8 + 1/16 + … <= 1
+ 1/4 + 1/8 + 1/16 + … <= 1/2
+ + 1/8 + 1/16 + … <= 1/4

$\sum j\cdot 2^{-j}$ <= 2

So $2^L \sum j\, 2^{-j} \leq 2\cdot 2^L = 2n$ O(n)

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$$2^L \sum_{j=1}^{\ell} j/2^j \leq 2^{L+1}$$

Where L is O(log n)

So, the number of swaps is $\leq$ O(n)

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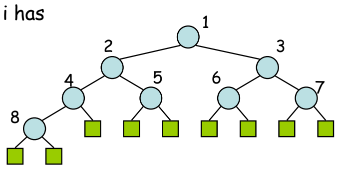
Implementing a Heap with an Array

A heap can be nicely represented by a vector (array-based), where the node at rank i has

- left child at rank 2i

and

- right child at rank 2i + 1



1	2	3	4	5	6	7	8

The leaves do no need to be explicitly stored

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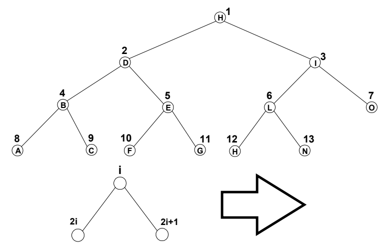
Example



1	2	3	4	5	6	7	8	9	10	11	12	13
H	D	I	B	E	L	O	A	C	F	G	H	N

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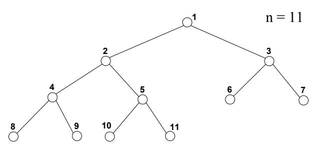
Reminder .....

Left child of T[i]	T[2i]	if	$2i \leq n$
Right child of T[i]	T[2i+1]	if	$2i + 1 \leq n$
Parent of T[i]	T[i div 2]	if	$i > 1$
The Root	T[1]	if	$T \neq 0$
Leaf? T[i]	TRUE	if	$2i > n$



n = 11

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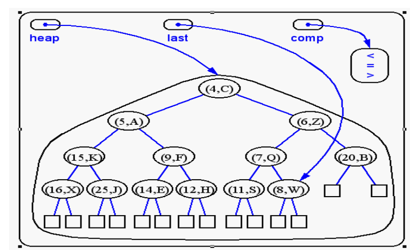
Implementation of a Priority Queue with a Heap



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(upheap)

insertItem	O(log n)
minKey, minElement	O(log 1)
removeMin	O(log n)

(remove root + downheap)

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Application: Sorting
Heap Sort

Construct initial heap O(n)

n times

remove root	O(1)
re-arrange	O(log n)
remove root	O(1)
re-arrange	O(log (n-1))
…	⋮
…	

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When there are i nodes left in the PQ: $\lfloor \log i \rfloor$

$\rightarrow$ TOT = $\sum_{i=1}^{n} \lfloor \log i \rfloor$

$= (n + 1)q - 2^{q+1} + 2$

where $q = \lfloor \log (n+1) \rfloor$

$\Rightarrow$ O(n log n)

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